M. Math 2008-2009 Advanced Functional Analysis Mid-term examination

22-09-2008 Time: 3hrs

Show complete work. Each question is worth 5 points

1) Let $X$ be a metrizable TVS and Y a TVS. Let $T : X \to Y$ be a linear map such that for every sequence $\{x_n\}_{n\geq 1} \subset X$, $x_n \to 0$ implies, $\{T(x_n)\}_{n\geq 1}$ is a cauchy sequence. Show that $T$ is continuous.

2) Let $X$ be a LCTVS such that every closed and bounded set is compact. Let $F \subset X$ be a totally bounded set. Show that the closed convex hull, $CO^{-}(\Gamma F)$ is a compact set, where $\Gamma$ is the unit circle.

3) Consider $\Lambda_m : \ell^2 \to C$ defined by $\Lambda_m(x) = \sum_1^m n^2 x(n)$. Let $x_n = \frac{1}{n} e_n$. Show that $K = \{x_n\}_{n\geq 1} \cup \{0\}$ is compact. Show that each $\Lambda_m(K)$ is a bounded but $\{\Lambda_m(K)\}_{m\geq 1}$ is not uniformly bounded.

4) Let $X$ be a LCTVS. Let $A \subset X$ be a balanced, closed convex set. Show that $(A^0) = A$.

5) Let $X$ be a TVS and $A, B$ two compact convex sets. Show that the convex hull $CO(A \cup B)$ is compact.

6) Let $\ell^1 = \{\{\alpha_n\}_{n\geq 1} \subset R : \sum |\alpha_n| < \infty\}$. Show that the extreme points of the closed unit ball are precisely $\{\pm e_n\}_{n\geq 1}$.

7) Let $K$ be a compact Hausdorff space. Let $C(K)_1^*$ be the closed unit ball of $C(K)^*$, equipped with the weak$^*$-topology. Let $\mathcal{P}$ denote the set of probability measures. Show that $C(K)_1^* = CO^{-}(\Gamma \mathcal{P})$, where the closure is taken in the weak$^*$-topology.

8) Show that $C([0,1])_1^*$ is not the norm-closed convex hull of its extreme points. You may assume that if $\mu_n \to \mu$ then $\mu_n(B) \to \mu(B)$ for all Borel sets $B$.